

THE EIGHTH MOMENT OF DIRICHLET L -FUNCTIONS

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ABSTRACT. We prove an asymptotic for the eighth moment of Dirichlet L -functions averaged over primitive characters χ modulo q , over all moduli $q \leq Q$ and with a short average on the critical line, conditionally on GRH. We derive the analogous result for the fourth moment of Dirichlet twists of $GL(2)$ L -functions. Our results match the moment conjectures in the literature; in particular, the constant 24024 appears as a factor in the leading order term of the eighth moment.

1. INTRODUCTION

There has been substantial and sustained research into moments of L -functions on the critical line. Much of this interest is generated by the presence of numerous applications, but moments are also studied for their own intrinsic interest. As well, understanding moments of L -functions involves developing tools which better elucidates the nature of the approximate orthogonality of the family of L -functions under consideration.

The first moments studied were those of the Riemann zeta function, which are averages of the form

$$I_k(T) := \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt.$$

Here, asymptotic formulae were proven for $k = 1$ by Hardy and Littlewood and for $k = 2$ by Ingham (see [19] VII). Despite extensive further work, including various refinements of the result of Ingham, no such result is available for any other values of k .

A well known folklore conjecture states that $I_k(T) \sim c_k T(\log T)^{k^2}$ for constants c_k depending on k . The values of c_k were mysterious for general k until the work of Keating and Snaith [10] which related these moments to circular unitary ensembles and provided precise conjectures for c_k . The choice of group is consistent with the Katz-Sarnak philosophy [9], which indicates that the symmetry group associated to this family should be unitary. Based on heuristics for shifted divisor sums, Conrey and Ghosh derived a conjecture in the case $k = 3$ [3] and Conrey and Gonek derived a conjecture in the case $k = 4$ [4]. Further conjectures including lower order terms, and for other symmetry groups are available from the work of Conrey, Farmer, Keating, Rubinstein and Snaith [2] as well as from the work of Diaconu, Goldfeld and Hoffstein [7].

In support of these conjectures, lower bounds of the the right order of magnitude are available due to Rudnick and Soundararajan [13], while good upper bounds are available conditionally on RH, due to Soundararajan, of the form $I_k(T) \ll T(\log T)^{k^2+\epsilon}$ [16].

The situation for other families of L -functions is very similar; asymptotics are only available for small values of k . While the asymptotic for the sixth moment of $\zeta(1/2 + it)$ remains out of reach, Conrey, Iwaniec and Soundararajan [5] have recently derived an asymptotic formula for the sixth moment of Dirichlet L -functions with a power saving error term. Instead of fixing the modulus q and only averaging over characters $\chi \pmod{q}$, they also average over the modulus $q \leq Q$, which gives them a larger family of size Q^2 . As well, they include a short average on the critical line. Since the family they consider

is also unitary, the asymptotic is very similar to the conjectured asymptotic for the sixth moment of $\zeta(1/2 + it)$.

To be more precise, we first introduce some notation. Let $\chi \pmod{q}$ be a primitive even Dirichlet character, and let (for $\text{Re } s > 1$),

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

be the L -function associated to it. Then the completed L -function $\Lambda(s, \chi)$ satisfies

$$\Lambda\left(\frac{1}{2} + s, \chi\right) = \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{1}{4} + \frac{s}{2}\right) L\left(\frac{1}{2} + s, \chi\right) = \epsilon_{\chi} \Lambda\left(\frac{1}{2} - s, \overline{\chi}\right),$$

where $|\epsilon_{\chi}| = 1$.

Let $\sum_{\chi \pmod{q}}^b$ indicate that a sum is over primitive even Dirichlet characters, and $\phi^b(q)$ denote the number of primitive even Dirichlet characters with modulus q . Then Corollary 1 in the work of Conrey, Iwaniec, and Soundararajan [5] states

$$\begin{aligned} & \sum_{q \leq Q} \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, \chi\right) \right|^6 dy \\ & \sim 42a_3 \sum_{q \leq Q} \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^5}{\left(1 + \frac{4}{p} + \frac{1}{p^2}\right)} \phi^b(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2 + iy}{2}\right) \right|^6 dy, \end{aligned}$$

where

$$a_3 = \prod_p \left(1 - \frac{1}{p^4}\right) \left(1 + \frac{4}{p} + \frac{1}{p^2}\right).$$

This is consistent with the conjecture in [2], and fits with the analogous conjecture for $\zeta(1/2 + it)$ in [4]. The authors of [5] later state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term. Note that the average over y is fairly short due to the rapid decay of the Γ function along vertical lines. However, deriving an analogous result without the average over y remains an open problem. Here, the restriction to even characters is a technical convenience, and the analogous result may be derived for odd characters using the same method.

In this paper, we shall derive asymptotics for the eighth moment of Dirichlet L -functions and for the fourth moment of Dirichlet twists of certain $GL(2)$ L -functions, conditionally on GRH.

From [2], we may derive the conjecture that as $q \rightarrow \infty$ with $q \not\equiv 2 \pmod{4}$,

$$\frac{1}{\phi^b(q)} \sum_{\chi \pmod{q}}^b |L(\tfrac{1}{2}, \chi)|^8 \sim 24024 a_4 \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \frac{(\log q)^{16}}{16!},$$

where

$$(1) \quad a_4 = \prod_p \left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right).$$

Towards this conjecture, we shall prove that

$$\begin{aligned}
& \sum_{q \leq Q} \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, \chi\right) \right|^8 dy \\
(2) \quad & \sim 24024 a_4 \sum_{q \leq Q} \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \phi^b(q) \frac{(\log q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2 + iy}{2}\right) \right|^8 dy
\end{aligned}$$

conditionally on GRH.

Our result (2) follows immediately from the following theorem.

Theorem 1. *Assume GRH. Let Ψ be a smooth function compactly supported in $[1, 2]$. Then, we have*

$$\begin{aligned}
& \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, \chi\right) \right|^8 dy \\
& = 24024 a_4 \sum_q \Psi\left(\frac{q}{Q}\right) \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \phi^b(q) \frac{(\log q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2 + iy}{2}\right) \right|^8 dy \\
& \quad + O(Q^2 (\log Q)^{15+\epsilon}).
\end{aligned}$$

Our method requires GRH to gain upper bounds on certain shifted moments, which allows us to reduce the length of the Dirichlet polynomials arising from the approximate functional equation (see Section 3). Indeed, the error in this truncation can be expressed through standard methods as a contour integral in which shifted L -functions appear. Such shifted moments for $\zeta(s)$ was studied on RH by V. Chandee in [1] and this method was used to compute the asymptotics for the second moment of $GL(2)$ forms twisted by quadratic characters on GRH by Soundararajan and Young [17]. The method for studying shifted moments is based on Soundararajan's work [16] which combined methods from Selberg's work [14] [15] with the observation that for the purposes of upper bounds on $\log \zeta(1/2 + it)$, the effect of zeros near $1/2 + it$ is benign.

In the proof of our result, the truncation of the Dirichlet polynomials allows us to switch to the complementary divisor and reduce the problem to one involving congruences with a smaller conductor (see Section 6.1). In extracting the main terms and bounding the error terms, we need to sum smoothly over the Dirichlet polynomials as well as over the modulus. For this, we introduce various Mellin transforms similar to the treatment in the work of Conrey, Iwaniec and Soundararajan [5] - this reduces the question to various contour integrals by standard methods.

Our method also allows us to compute the fourth moment of Dirichlet twists of a $GL(2)$ automorphic L -function. To be precise, let f be a holomorphic modular form of weight k and full level¹. We write

$$L(s, f) = \sum_n \frac{a(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

¹Our methods apply to other $GL(2)$ forms; we restrict our attention to this case to ease notation.

normalized so that the critical line is at $\text{Re } s = 1/2$. Then the twisted L -function $L(s, f \times \chi)$ has Dirichlet series

$$L(s, f \times \chi) = \sum_n \frac{a(n)\chi(n)}{n^s},$$

and satisfies the functional equation

$$(3) \quad \Lambda\left(\frac{1}{2} + s, f \times \chi\right) := \left(\frac{q}{2\pi}\right)^s \Gamma\left(s + \frac{k}{2}\right) L\left(s + \frac{1}{2}, f \times \chi\right) = \omega_\chi \Lambda\left(\frac{1}{2} - s, f \times \bar{\chi}\right),$$

where $\omega_\chi = i^k \frac{\tau(\chi)^2}{q}$. Here, $\tau(\chi)$ is the Gauss sum and so $|\omega_\chi| = 1$. Now let

$$f_2 := \prod_p \left(1 - \frac{a(p)^2 - 2}{p} + \frac{1}{p^2}\right)^{-3} \left(1 + \frac{a(p)^2 + 2}{p} - \frac{4a(p)^2 - 2}{p^2} + \frac{a(p)^2 + 2}{p^3} + \frac{1}{p^4}\right),$$

and $B_p(f, 1/2)$ be defined by

$$(4) \quad \left(1 - \frac{1}{p}\right)^{-4} \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{a(p)^2 - 2}{p} + \frac{1}{p^2}\right)^{-3} \left(1 + \frac{a(p)^2 + 2}{p} - \frac{4a(p)^2 - 2}{p^2} + \frac{a(p)^2 + 2}{p^3} + \frac{1}{p^4}\right).$$

Then we have the following

Theorem 2. *Assume GRH. Let Ψ be a smooth function compactly supported in $[1, 2]$. Then, we have*

$$\begin{aligned} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^* \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, f \times \chi\right) \right|^4 dy \\ = \frac{1}{2\pi^2} f_2 \sum_q \Psi\left(\frac{q}{Q}\right) \phi^*(q) (2 \log q)^4 \prod_{p|q} \frac{1}{B_p(f, 1/2)} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{k/2 + iy}{2}\right) \right|^4 dy \\ + O(Q^2 (\log Q)^{3+\epsilon}). \end{aligned}$$

This is consistent with the conjectures for the fourth moment of a unitary family of L -functions, as in [2]. The proof of Theorem 2 is similar to that of Theorem 1. In the sequel, we will focus on proving Theorem 1 and indicate how to modify the proof for Theorem 2 in Section 12.

2. NOTATION AND PRELIMINARY LEMMAS

In this section, we introduce some notation and record some standard results. Define

$$\Lambda(s, \chi; t) := \Lambda^4(s + it, \chi) \Lambda^4(s - it, \bar{\chi}),$$

and

$$G(s, t) := \Gamma^4\left(\frac{s}{2} + \frac{it}{2}\right) \Gamma^4\left(\frac{s}{2} - \frac{it}{2}\right).$$

We then have

$$\Lambda\left(\frac{1}{2}, \chi; t\right) = G\left(\frac{1}{2}, t\right) L^4\left(\frac{1}{2} + it, \chi\right) L^4\left(\frac{1}{2} - it, \bar{\chi}\right).$$

Let $\tau_4(n) = \sum_{n=n_1 n_2 n_3 n_4} 1$. Therefore for $\text{Re } (s) > 1$, we write

$$L^4(s + it, \chi) L^4(s - it, \bar{\chi}) = \sum_{m, n=1}^{\infty} \frac{\tau_4(m) \tau_4(n)}{m^s n^s} \chi(m) \bar{\chi}(n) \left(\frac{n}{m}\right)^{it}.$$

Moreover, define

$$(5) \quad W(x, t) := \frac{1}{2\pi i} \int_{(1)} G(1/2 + s, t) x^{-s} \frac{ds}{s},$$

$$P(\chi, t) := \sum_{m, n=1}^{\infty} \frac{\tau_4(m) \tau_4(n) \chi(m) \overline{\chi}(n)}{\sqrt{mn}} W\left(\frac{mn\pi^4}{q^4}, t\right),$$

$$(6) \quad V(\xi, \eta; \mu) = \int_{-\infty}^{\infty} \left(\frac{\eta}{\xi}\right)^{it} W\left(\frac{\xi\eta\pi^4}{\mu^4}, t\right) dt,$$

and

$$\Lambda_1(\chi) = \sum_{m, n=1}^{\infty} \frac{\tau_4(m) \tau_4(n)}{\sqrt{mn}} \chi(m) \overline{\chi}(n) V(m, n; q).$$

Our first Lemma is an approximate functional equation.

Lemma 1. *With notation as above, we have*

$$(7) \quad \Lambda(1/2, \chi; t) = 2P(\chi, t),$$

and

$$(8) \quad \int_{-\infty}^{\infty} \Lambda(1/2, \chi; t) dt = 2\Lambda_1(\chi).$$

Proof. We begin by writing

$$P(\chi, t) = \frac{1}{2\pi i} \int_{(1)} \Lambda(1/2 + s, \chi; t) \frac{ds}{s}.$$

Shifting contours of integration to $\operatorname{Re} s = -1$, we pass a simple pole at $s = 0$ with residue $\Lambda(1/2, \chi; t)$. Upon applying the functional equation, the remaining integral is

$$\frac{1}{2\pi i} \int_{(-1)} \Lambda(1/2 - s, \chi; t) \frac{ds}{s} = -\frac{1}{2\pi i} \int_{(1)} \Lambda(1/2 + s, \chi; t) \frac{ds}{s} = -P(\chi, t).$$

This proves (7). Integrating both sides of (7) gives (8). \square

The integration of $\Lambda(1/2, \chi; t)$ in t is needed to restrict the range of m, n . From the following lemma, we see that the main term of $\Lambda_1(\chi)$ comes from when m, n are both at most $q^{2+\epsilon}$.

Lemma 2. *The weight function $W(x, t)$ defined in (5) is a smooth function of $x \in (0, \infty)$ and moreover, for any $x > 1$ and any non-negative integer ν ,*

$$W^{(\nu)}(x, t) \ll_{\nu} \exp(-c_0 x^{1/4})$$

for some absolute constant $c_0 > 0$. This in term gives us that the function V defined in (6) satisfies

$$V(\xi, \eta; \mu) \ll \exp\left(-c_1 \left(\frac{\max(\xi, \eta)^2}{\mu^4}\right)^{1/4}\right).$$

Proof. This lemma is the same as Lemma 1 in [5]. \square

Next, we introduce some notation which will be used when calculating the arithmetic factor a_4 in (1). Let

$$\mathcal{B}_p(s) = \sum_{r=0}^{\infty} \frac{\tau_4^2(p^r)}{p^{2rs}}, \quad \mathcal{B}_q(s) = \prod_{p|q} B_p(s),$$

and

$$A_p(s) = \left(1 - \frac{1}{p^{2s}}\right)^{16} B_p(s), \quad A(s) = \prod_p A_p(s).$$

Lemma 3. *With notation as above, we have that for $\operatorname{Re}(s) > 1/2$,*

$$B_p(s) = \left(1 - \frac{1}{p^{2s}}\right)^{-7} \left(1 + \frac{9}{p^{2s}} + \frac{9}{p^{4s}} + \frac{1}{p^{6s}}\right),$$

and

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_4^2(n)}{n^{2s}} = \frac{\zeta^{16}(2s)A(s)}{\mathcal{B}_q(s)},$$

where $A(s)$ converges absolutely when $\operatorname{Re}(s) > 1/4$.

Proof. This is a standard proof which involves writing out both sides in terms of Euler products. See Section 2 of [2] for similar proofs. \square

The next lemma is required to compute the main term of the fourth moment of Dirichlet twists of a $GL(2)$ automorphic L -function. We define

$$\mathcal{B}_p(f, s) = \sum_{r=0}^{\infty} \frac{\sigma_f^2(p^r)}{p^{2rs}}, \quad \mathcal{B}_q(f, s) = \prod_{p|q} B_p(f, s),$$

and

$$A(f, s) = \prod_p \left(1 - \frac{1}{p^{2s}}\right)^4 B_p(f, s).$$

Lemma 4. *With notation as above, we have for $\operatorname{Re}(s) > 1/2$,*

$$(9) \quad \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\sigma_f^2(n)}{n^{2s}} = \frac{\zeta^4(2s)A(f, s)}{\mathcal{B}_q(f, s)},$$

where $B_p(f, 1/2)$ is as in (4) and

$$A_p(f, 1/2) = \frac{6}{\pi^2} \prod_p \left(1 - \frac{a(p)^2 - 2}{p} + \frac{1}{p^2}\right)^{-3} \left(1 + \frac{a(p)^2 + 2}{p} - \frac{4a(p)^2 + 2}{p^2} + \frac{a(p)^2 + 2}{p^3} + \frac{1}{p^4}\right),$$

converges absolutely when $\operatorname{Re}(s) > 1/4$.

Proof. Let \oint denote the contour integral around the unit circle. The proof of (9) can be found in [2], where $B_p(f, 1/2)$ is given by

$$\begin{aligned} B_p(f, 1/2) &= \int_0^1 \left(1 - \frac{\alpha_p e(\theta)}{p^{1/2}}\right)^{-2} \left(1 - \frac{\beta_p e(\theta)}{p^{1/2}}\right)^{-2} \left(1 - \frac{\alpha_p e(-\theta)}{p^{1/2}}\right)^{-2} \left(1 - \frac{\beta_p e(-\theta)}{p^{1/2}}\right)^{-2} d\theta \\ &= \frac{p^2}{2\pi i} \oint z^3 \frac{\left(z - \frac{p^{1/2}}{\alpha_p}\right)^{-2} \left(z - \frac{p^{1/2}}{\beta_p}\right)^{-2}}{\left(z - \frac{\alpha_p}{p^{1/2}}\right)^2 \left(z - \frac{\beta_p}{p^{1/2}}\right)^2} dz \\ &= p^2 \left(\operatorname{Res}_{z=\frac{\alpha_p}{p^{1/2}}} + \operatorname{Res}_{z=\frac{\beta_p}{p^{1/2}}} \right) z^3 \frac{\left(z - \frac{p^{1/2}}{\alpha_p}\right)^{-2} \left(z - \frac{p^{1/2}}{\beta_p}\right)^{-2}}{\left(z - \frac{\alpha_p}{p^{1/2}}\right)^2 \left(z - \frac{\beta_p}{p^{1/2}}\right)^2}. \end{aligned}$$

Using that $\alpha_p \beta_p = 1$, the sum of the residues is

$$\begin{aligned} \frac{1}{p^6} \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{-3} \left(1 - \frac{\alpha_p^2}{p}\right)^{-3} \left(1 - \frac{\beta_p^2}{p}\right)^{-3} \times \\ \times [(1 + 2p + 2p^2 + 2p^3 + p^4) + (\alpha_p + \beta_p)^2(p - 4p^2 + p^3)], \end{aligned}$$

where gives the desired form for $B_p(f, 1/2)$ upon using that $\alpha_p + \beta_p = a(p)$. \square

We also need orthogonality relations for characters.

Lemma 5. *If m, n are integers with $(mn, q) = 1$ then*

$$\sum_{\chi \pmod{q}}^* \chi(m) \overline{\chi}(n) = \sum_{\substack{q=dr \\ r|(m-n)}} \mu(d) \phi(r),$$

and

$$\sum_{\chi \pmod{q}}^b \chi(m) \overline{\chi}(n) = \frac{1}{2} \sum_{\substack{q=dr \\ r|(m \pm n)}} \mu(d) \phi(r).$$

Proof. The first claim follows from the orthogonality of all characters and Mobius inversion, while the second claim follows from the first by detecting even characters with $\frac{1+\chi(-1)}{2}$. \square

Lemma 6. *Let y, t and S be real numbers and q and k be natural numbers with $y^k \leq \frac{\sqrt{\phi(q)S}}{\log qS}$. For any complex numbers $a(p)$, we have that*

$$\sum_{\chi \pmod{q}} \int_S^{2S} \left| \sum_{p \leq y} \frac{a(p) \chi(p)}{p^{1/2+it}} \right|^{2k} dt \ll \phi(q) S k! \left(\sum_{p \leq y} \frac{|a(p)|^2}{p} \right)^k,$$

where the implied constant is absolute.

Proof. Write

$$\left(\sum_{p \leq y} \frac{a(p) \chi(p)}{p^{1/2+it}} \right)^k = \sum_{n \leq y^k} \frac{a_{k,y}(n) \chi(n)}{n^{1/2+it}},$$

where $a_{k,y}(n) = \sum_{\substack{p_1 \dots p_k = n \\ p_i \leq y}} a(p_1) \dots a(p_k)$. Then by orthogonality of Dirichlet characters

$$\begin{aligned} \sum_{\chi \pmod{q}} \int_S^{2S} \left| \sum_{p \leq y} \frac{a(p)\chi(p)}{p^{1/2+it}} \right|^{2k} dt &= \phi(q) \sum_{\substack{m, n \leq y^k \\ (mn, q)=1 \\ q|m-n}} \frac{a_{k,y}(m)\overline{a_{k,y}(n)}}{\sqrt{mn}} \int_S^{2S} \left(\frac{n}{m}\right)^{it} dt \\ &= \phi(q)S \sum_{\substack{n \leq y^k \\ (n, q)=1}} \frac{|a_{k,y}(n)|^2}{n} + O \left(\phi(q) \sum_{\substack{m, n \leq y^k \\ m \neq n, q|m-n}} \frac{|a_{k,y}(m)\overline{a_{k,y}(n)}|}{\sqrt{mn} |\log(m/n)|} \right). \end{aligned}$$

The sum in the diagonal term is bounded by

$$\begin{aligned} &\leq \sum_{n \leq y^k} \frac{|a_{k,y}(n)|^2}{n} = \sum_{p_1, \dots, p_r \leq y} \sum_{\substack{\alpha_1, \dots, \alpha_r \geq 1 \\ \sum \alpha_i = k}} \binom{k}{\alpha_1, \dots, \alpha_r}^2 \frac{|a_{p_1}|^{2\alpha_1} \dots |a_{p_r}|^{2\alpha_r}}{p_1^{\alpha_1} \dots p_r^{\alpha_r}} \\ &\leq k! \sum_{p_1, \dots, p_r \leq y} \sum_{\substack{\alpha_1, \dots, \alpha_r \geq 1 \\ \sum \alpha_i = k}} \binom{k}{\alpha_1, \dots, \alpha_r} \frac{|a_{p_1}|^{2\alpha_1} \dots |a_{p_r}|^{2\alpha_r}}{p_1^{\alpha_1} \dots p_r^{\alpha_r}} \\ &= \left(\sum_{p \leq y} \frac{|a(p)|^2}{p} \right)^k. \end{aligned}$$

Now we return to the error term. Since $y^k \leq \frac{\sqrt{\phi(q)S}}{\log qS}$, either $y^k \leq \frac{\phi(q)}{\log qS}$ or $y^k \leq \frac{S}{\log qS}$.

If $y^k \leq \frac{\phi(q)}{\log qS}$, then $q|m-n$ if and only if $m=n$ so there are no additional terms. If $y^k \leq \frac{S}{\log qS}$, then the error term is bounded by

$$\begin{aligned} &\ll \phi(q) \sum_{m \leq y^k} \frac{|a_{k,y}(m)|^2}{m} \sum_{\substack{n \neq m \\ n \leq y^k}} \frac{1}{|\log(m/n)|} \ll \phi(q) y^k \log(y^k) \sum_{m \leq y^k} \frac{|a_{k,y}(m)|^2}{m} \\ &\ll \phi(q) S k! \left(\sum_{p \leq y} \frac{|a(p)|^2}{p} \right)^k. \end{aligned}$$

□

Finally, the following propositions estimate the moments and shifted moments of Dirichlet L -functions and Dirichlet twists of a $GL(2)$ automorphic L -function. They are crucial tools in truncating the length of the polynomials in Section 3 and hence in bounding off-diagonal terms in Section 7.

Proposition 1. *Assume GRH. For $q \geq 3$, $S > 0$ and for any positive real number k and any $\epsilon > 0$ we have*

$$\sum_{\chi \pmod{q}} \int_{S \leq |s| < 2S} \left| L\left(\frac{1}{2} + s, \chi\right) \right|^{2k} ds \ll \phi(q) S (\log q(S+1))^{k^2+\epsilon},$$

where the integration over s is taken to be on a vertical line with $0 \leq \operatorname{Re}(s) \leq \frac{1}{\log Q}$.

Proof. To prove the Theorem, it is enough to show that

$$(10) \quad \sum_{\chi \pmod{q}}^* \int_{S \leq |s| < 2S} \left| L\left(\frac{1}{2} + s, \chi\right) \right|^{2k} ds \ll \phi(q) S (\log q S)^{k^2 + \epsilon}$$

since we can deduce the result for sum over all characters as the following.

$$\begin{aligned} & \sum_{\chi \pmod{q}} \int_{S \leq |s| \leq 2S} \left| L\left(\frac{1}{2} + s, \chi\right) \right|^{2k} ds \\ & \leq \sum_{q_1 | q} \sum_{\chi_1 \pmod{q_1}}^* \left(\int_{S \leq |s| \leq 2S} \left| L\left(\frac{1}{2} + s, \chi_1\right) \right|^{2k} ds \right) \prod_{\substack{p|q \\ p \nmid q_1}} \left(1 + \frac{1}{p^{1/2}} \right)^{2k} \\ & \ll (\log q S)^{k^2 + \epsilon} \sum_{q_1 | q} \phi(q_1) \prod_{\substack{p|q \\ p \nmid q_1}} \left(1 + \frac{1}{p^{1/2}} \right)^{2k} \\ & \ll q (\log q)^{k^2 + \epsilon} \ll \phi(q) (\log q)^{k^2 + \epsilon}, \end{aligned}$$

where the first inequality comes from

$$\begin{aligned} \sum_{q_1 | q} \phi(q_1) \prod_{\substack{p|q \\ p \nmid q_1}} \left(1 + \frac{1}{p^{1/2}} \right)^{4k} &= \prod_{p^e || q} \left[\left(1 + \frac{1}{p^{1/2}} \right)^{4k} + \phi(p) + \phi(p^2) + \dots + \phi(p^e) \right] \\ &= \prod_{p^e || q} p^e \left(1 + O\left(\frac{1}{p^{3/2}}\right) \right) \ll q, \end{aligned}$$

and the last inequality comes from $\phi(q) \gg \frac{q}{\log \log q}$.

The proof of (10) is carried by the same arguments as the proof of Corollary A in [16], but here we use the orthogonality relation in Lemma 5 instead of Lemma 3 in [16]. \square

Similarly we have

Proposition 2. *Assume GRH. For $q \geq 3$, $S > 0$ and for any positive real number k and any $\epsilon > 0$ we have*

$$\sum_{\chi \pmod{q}} \int_{S \leq |s| < 2S} \left| L\left(\frac{1}{2} + s, f \times \chi\right) \right|^{2k} \ll \phi(q) S (\log q (S + 1))^{k^2 + \epsilon},$$

where the integration over s is taken to be on a vertical line with $0 \leq \operatorname{Re}(s) \leq \frac{1}{\log Q}$.

Next, we have the propositions for shifted moments.

Proposition 3. *Let q be large and let z_1, z_2 be complex numbers with $0 \leq \operatorname{Re}(z_i) \leq 1/\log q$ and with $|z_i| \leq q$. Assume GRH. Then for any positive real number k and any $\epsilon > 0$ we have*

$$\sum_{\chi \pmod{q}}^b \left| L\left(\frac{1}{2} + z_1, \chi\right) \right|^{2k} \left| L\left(\frac{1}{2} + z_2, \chi\right) \right|^{2k} \ll \phi(q) (\log q)^{2k^2 + \epsilon} \left(\min \left\{ (\log q)^{2k^2}, \frac{1}{|z_1 + \bar{z}_2|^{2k^2}} \right\} \right).$$

Proposition 4. *Let q be large and let z_1, z_2 be complex numbers with $0 \leq \operatorname{Re}(z_i) \leq 1/\log q$ and with $|z_i| \leq q$. Assume GRH. Then for any positive real number k and any $\epsilon > 0$ we have*

$$\sum_{\chi \pmod{q}}^* \left| L\left(\frac{1}{2} + z_1, f \times \chi\right) \right|^{2k} \left| L\left(\frac{1}{2} + z_2, f \times \chi\right) \right|^{2k} \\ \ll \phi(q)(\log q)^{2k^2+\epsilon} \left(\min \left\{ (\log q)^{2k^2}, \frac{1}{|z_1 + \bar{z}_2|^{2k^2}} \right\} \right).$$

The argument of the proof of both Propositions is similar to the proof of Theorem 1.1 in [1], which follows the method of [16] to obtain upper bounds of shifted moments of the Riemann zeta function. Here we will prove Proposition 3 in Section 11, and the proof of Proposition 4 can be proceeded in the same manner.

3. TRUNCATION

From Lemma 1, we want to study the moment

$$\mathcal{M} = \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, \chi\right) \right|^8 dy = 2\Delta(\Psi, Q),$$

where

$$\Delta(\Psi, Q) = \sum_q \sum_{\chi \pmod{q}}^b \Psi\left(\frac{q}{Q}\right) \Lambda_1(\chi).$$

By orthogonality as in Lemma 5, we obtain that

$$\Delta(\Psi, Q) = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d,r \\ (dr, mn)=1 \\ r|m \pm n}} \phi(r)\mu(d)\Psi\left(\frac{dr}{Q}\right) V(m, n, dr).$$

The first step to prove Theorem 1 is to truncate the sum $\Delta(\Psi, Q)$. For a fixed α , define

$$\tilde{\Delta}(\Psi, Q) = \sum_q \sum_{\chi \pmod{q}}^b \Psi\left(\frac{q}{Q}\right) \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \chi(m)\bar{\chi}(n) V\left(m, n; \frac{q}{(\log Q)^\alpha}\right).$$

Note that we expect (and will later show) that $\Delta(\Psi, Q) \asymp Q^2(\log Q)^{16}$. In this section, we prove that $\tilde{\Delta}(\Psi, Q)$ is a sufficiently close approximation to $\Delta(\Psi, Q)$. This will allow us to apply the complementary divisor trick to $\tilde{\Delta}$ and reduce the conductor in Section 6.

Proposition 5. *Let A be a fixed constant. Then for any $\epsilon > 0$,*

$$\Delta(\Psi, Q) - \tilde{\Delta}(\Psi, Q) \ll Q^2(\log Q)^{15+\epsilon},$$

where the implied constant depends on ϵ and P .

The key ingredient for proving the proposition is Proposition 3 for $k = 2$.

Proof. We have that $\Delta(\Psi, Q) - \tilde{\Delta}(\Psi, Q)$ is

$$\begin{aligned} & \sum_q \Psi\left(\frac{q}{Q}\right) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{(1)} \sum_{\chi \bmod q} \sum_{m,n}^b \frac{\tau_4(m)\chi(m)}{m^{1/2+s+it}} \frac{\tau_4(n)\overline{\chi(n)}}{n^{1/2+s-it}} G\left(\frac{1}{2} + s, t\right) \\ & \quad \cdot \left[\left(\frac{q}{\pi}\right)^{4s} - \left(\frac{q}{\pi(\log Q)^\alpha}\right)^{4s} \right] \frac{ds}{s} dt \\ & = \sum_q \Psi\left(\frac{q}{Q}\right) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{(1)} G\left(\frac{1}{2} + s, t\right) \left[\left(\frac{q}{\pi}\right)^{4s} - \left(\frac{q}{\pi(\log Q)^\alpha}\right)^{4s} \right] \\ & \quad \cdot \sum_{\chi \bmod q}^b L^4\left(\frac{1}{2} + s + it, \chi\right) L^4\left(\frac{1}{2} + s - it, \bar{\chi}\right) \frac{ds}{s} dt. \end{aligned}$$

Note that the integrand is holomorphic. Now we move the line of integration to $\operatorname{Re}(s) = 0$ and write $s = it_1$. Then

$$\left[\left(\frac{q}{\pi}\right)^{4s} - \left(\frac{q}{\pi(\log Q)^\alpha}\right)^{4s} \right] \frac{1}{s} \ll \log \log Q.$$

Moreover, by Stirling's formula we have

$$G\left(\frac{1}{2} + s, t\right) \ll \exp(-|t_1 + t| - |t_1 - t|).$$

By changing variables $t_1 + t = u_1$ and $t - t_1 = u_2$ and using the above bounds, we have that

$$\begin{aligned} & \Delta(\Psi, Q) - \tilde{\Delta}(\Psi, Q) \\ & \ll \log \log Q \sum_q \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-|u_1| - |u_2|) \\ & \quad \sum_{\chi \bmod q}^b \left| L\left(\frac{1}{2} + iu_1, \chi\right) \right|^4 \left| L\left(\frac{1}{2} + iu_2, \chi\right) \right|^4 du_1 du_2 \end{aligned}$$

If both $|u_1|, |u_2| \leq q$, then we use Proposition 3 to bound the sum over $\chi \bmod q$ and obtain that the contribution is $\ll Q^2(\log Q)^{15+\epsilon}$.

For larger u_1, u_2 , we apply the Lindelöf bound to see that

$$\begin{aligned} & \sum_{\chi \bmod q}^b \left| L\left(\frac{1}{2} + iu_1, \chi\right) \right|^4 \left| L\left(\frac{1}{2} + iu_2, \chi\right) \right|^4 \\ & \ll Q^{1+\epsilon} ((|u_1| + 1)(|u_2| + 1))^\epsilon. \end{aligned}$$

Hence when either $|u_1|$ or $|u_2|$ is greater than q , the contribution is $\ll Q^{-100}$. This proves the proposition. \square

4. SPLITTING OFF THE DIAGONAL TERMS

From the previous section, it is sufficient to consider

$$\tilde{\Delta}(\Psi, Q) = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d,r \\ (dr, mn)=1 \\ r|m \pm n}} \phi(r)\mu(d)\Psi\left(\frac{dr}{Q}\right) V\left(m, n, \frac{dr}{(\log Q)^\alpha}\right).$$

Let $D = (\log Q)^\delta$ for $\delta > 0$ a parameter to be determined (eventually we pick $\delta = 130$) and split

$$(11) \quad \tilde{\Delta}(\Psi, Q) = \mathcal{D}(\Psi, Q) + \mathcal{S}(\Psi, Q) + \mathcal{G}(\Psi, Q)$$

where the diagonal term $\mathcal{D}(\Psi, Q)$ consists of the terms with $m = n$, the term $\mathcal{S}(\Psi, Q)$ consists of the remaining terms with $d > D$ and $\mathcal{G}(\Psi, Q)$ consists of the rest of the terms with $d < D$.

4.1. The diagonal terms $\mathcal{D}(\Psi, Q)$. We have that

$$\begin{aligned} \mathcal{D}(\Psi, Q) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\tau_4^2(n)}{n} \sum_{\substack{d, r \\ (dr, n)=1}} \phi(r) \mu(d) \Psi \left(\frac{dr}{Q} \right) V \left(n, n, \frac{dr}{(\log Q)^\alpha} \right) \\ &= \sum_q \phi^b(q) \Psi \left(\frac{q}{Q} \right) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{(1)} \left(\sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\tau_4^2(n)}{n^{1+2s}} \right) G(1/2 + s, t) \left(\frac{q}{2\pi(\log Q)^\alpha} \right)^{4s} \frac{ds}{s} dt. \end{aligned}$$

From Lemma 3,

$$\sum_{n=1}^{\infty} \frac{\tau_4^2(n)}{n^{1+2s}} = \zeta^{16}(1+2s) \frac{A(s+1/2)}{\mathcal{B}_q(s+1/2)},$$

where A is analytic when $\operatorname{Re} s > -1/4$.

Thus the integrand has a pole of order 17 at $s = 0$ in the region $\operatorname{Re} s > -1/4 + \epsilon$, and shifting s to $\operatorname{Re}(s) = -1/4 + \epsilon$ gives a residue of

$$\frac{4^{16}}{2^{16} 16!} \frac{A(1/2)}{B_q(1/2)} G(1/2, t) (\log q)^{16} + O((\log q)^{15+\epsilon}).$$

Hence

$$\mathcal{D}(\Psi, Q) = 2^{16} \sum_q \phi^b(q) \Psi \left(\frac{q}{Q} \right) \frac{(\log q)^{16}}{16!} \frac{A(1/2)}{B_q(1/2)} \int_{-\infty}^{\infty} G(1/2, t) dt + O(Q^2 (\log Q)^{15+\epsilon}).$$

Since $\phi^b(q) = \frac{1}{2} \phi^*(q) + O(1)$, and the function ϕ^* is multiplicative with $\phi^*(p) = p - 2$ and $\phi^*(p) = p^{k-2}(p-1)^2$ for $k \geq 2$, we obtain that the main term of $\mathcal{D}(\Psi, Q)$ is

$$(12) \quad 2^{16} Q^2 \frac{(\log Q)^{16}}{16!} \tilde{\Psi}(2) \frac{A(1/2)}{2} \prod_p \left(1 - \frac{1}{p} \right) \left(1 + \frac{1}{B_p(1/2)} \left(\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3} \right) \right) \int_{-\infty}^{\infty} G(1/2, t) dt,$$

where

$$(13) \quad \tilde{\Psi}(s) = \int_0^\infty \Psi(u) u^s \frac{du}{u}$$

is the Mellin transform of Ψ .

5. THE OFF-DIAGONAL TERM: THE SUM $\mathcal{S}(\Psi, Q)$

Recall that

$$\mathcal{S}(\Psi, Q) = \frac{1}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m) \tau_4(n)}{\sqrt{mn}} \sum_{\substack{d, r \\ (dr, mn)=1 \\ r|m \pm n \\ d > D}} \phi(r) \mu(d) \Psi \left(\frac{dr}{Q} \right) V \left(m, n, \frac{dr}{(\log Q)^\alpha} \right).$$

We express the condition $r|m \pm n$ using the even characters $\chi \pmod{r}$. Specifically,

$$\mathcal{S}(\Psi, Q) = \sum_{\substack{d, r \\ d > D}} \mu(d) \Psi\left(\frac{dr}{Q}\right) \sum_{\substack{\chi \pmod{r} \\ \chi(-1)=1}} \sum_{\substack{m, n=1 \\ m \neq n \\ (d, mn)=1}}^{\infty} \frac{\chi(m) \bar{\chi}(n) \tau_4(m) \tau_4(n)}{\sqrt{mn}} V\left(m, n, \frac{dr}{(\log Q)^\alpha}\right).$$

The contribution of the principal character $\chi = \chi_0$ gives a main term, and the non-principal characters contribute to an acceptable error term.

Proposition 6. *We have*

$$\mathcal{S}(\Psi, Q) = \mathcal{MS}(\Psi, Q) + O\left(\frac{Q^2(\log Q)^{16+\epsilon}}{D^{1-\epsilon}}\right),$$

where

$$\mathcal{MS}(\Psi, Q) = - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m) \tau_4(n)}{\sqrt{mn}} \sum_{(q, mn)=1} \Psi\left(\frac{q}{Q}\right) \left(\sum_{\substack{dr=q \\ d \leq D}} \mu(d) \right) V\left(m, n, \frac{q}{(\log Q)^\alpha}\right).$$

Note that taking $D > \log Q$ gives us an acceptable error term.

Proof. The principal character gives

$$\sum_q \left(\sum_{\substack{dr=q \\ d > D}} \mu(d) \right) \Psi\left(\frac{q}{Q}\right) \sum_{\substack{m, n=1 \\ m \neq n \\ (q, mn)=1}}^{\infty} \frac{\tau_4(m) \tau_4(n)}{\sqrt{mn}} V\left(m, n, \frac{q}{(\log Q)^\alpha}\right).$$

Since $\sum_{dr=q} \mu(d) = 0$ for $q > 1$, the above equals to \mathcal{MS} stated in the proposition.

Now we consider the contribution of the non-principal characters. We first reintroduce the terms $m = n$. This gives an acceptable error since

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\tau_4^2(n)}{n} \sum_{\substack{d, r \\ (dr, n)=1 \\ d > D}} \phi(r) \mu(d) \Psi\left(\frac{dr}{Q}\right) V\left(n, n, \frac{dr}{(\log Q)^\alpha}\right) \\ &= \sum_{\substack{d, r \\ d > D}} \phi(r) \mu(d) \Psi\left(\frac{dr}{Q}\right) \frac{1}{2\pi i} \int_{(1)} \int_{-\infty}^{\infty} \sum_{\substack{n=1 \\ (dr, n)=1}}^{\infty} \frac{\tau_4^2(n)}{n^{1+2s}} G(1/2 + s, t) \left(\frac{dr}{2\pi(\log Q)^\alpha}\right)^{4s} \frac{ds}{s} dt. \end{aligned}$$

From the calculation of the diagonal term, the double integral is $\ll (\log dr)^{16}$. Therefore the above is

$$\ll (\log Q)^{16} \sum_{\substack{d, r \\ d > D}} \phi(r) \Psi\left(\frac{dr}{Q}\right) \ll (\log Q)^{16} \sum_{d > D} \sum_{r < 2Q/d} r \ll \frac{Q^2(\log Q)^{16}}{D}.$$

Now we bound the resultant sum

$$\begin{aligned}
& \sum_{\substack{d,r \\ d>D}} \mu(d) \Psi\left(\frac{dr}{Q}\right) \sum_{\substack{\chi \pmod{r} \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \sum_{\substack{m,n=1 \\ (d,mn)=1}}^{\infty} \frac{\chi(m) \bar{\chi}(n) \tau_4(m) \tau_4(n)}{\sqrt{mn}} V\left(m, n, \frac{dr}{(\log Q)^\alpha}\right) \\
&= \sum_{\substack{d,r \\ d>D}} \mu(d) \Psi\left(\frac{dr}{Q}\right) \sum_{\substack{\chi \pmod{r} \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \frac{1}{2\pi i} \int_{(1)} \int_{-\infty}^{\infty} \sum_{\substack{m,n=1 \\ (d,mn)=1}}^{\infty} \frac{\chi(m) \bar{\chi}(n) \tau_4(m) \tau_4(n)}{m^{1/2+s+it} n^{1/2+s-it}} \\
&\quad G(1/2+s, t) \left(\frac{dr}{2\pi(\log Q)^{100}}\right)^{4s} \frac{ds}{s} dt. \\
&= \mathcal{S}_1(\Psi, Q) + \mathcal{S}_2(\Psi, Q),
\end{aligned}$$

where $\mathcal{S}_1(\Psi, Q)$ is the sum over d, r such that $d > \sqrt{Q}$ and $r \leq \frac{2Q}{d}$, and $\mathcal{S}_2(\Psi, Q)$ is the sum over d and r such that $D < d \leq \sqrt{Q}$, so $\sqrt{Q} < r \leq \frac{2Q}{d}$. We have

$$\sum_{\substack{m,n=1 \\ (d,mn)=1}}^{\infty} \frac{\chi(m) \bar{\chi}(n) \tau_4(m) \tau_4(n)}{m^{1/2+s+it} n^{1/2+s-it}} = \frac{L^4(1/2+s+it, \chi) L^4(1/2+s-it, \bar{\chi})}{L_d^4(1/2+s+it, \chi) L_d^4(1/2+s-it, \bar{\chi})},$$

where $L_d(s, \psi) = \prod_{p|d} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1} \ll d^\epsilon$. Since χ is non-principal, we can shift the line of integration to $\operatorname{Re}(s) = \frac{1}{\log Q}$ without passing any poles. We write $s = \frac{1}{\log Q} + it_1$. We have seen in the previous section that

$$G(1/2+s, t) \ll \exp(-|t+t_1| - |t-t_1|) \ll \exp(-|t| - |t_1|).$$

Assuming GRH, we have

$$L(1/2+s \pm it, \chi) \ll q^\epsilon (1 + |t| + |t_1|)^\epsilon.$$

Therefore,

$$\mathcal{S}_1(\Psi, Q) \ll Q^\epsilon \sum_{d>\sqrt{Q}} \sum_{r \leq \frac{2Q}{d}} \phi(r) \ll Q^{3/2+\epsilon}.$$

Also, using Proposition 3 and the same argument as in Section 3 to bound $\mathcal{S}_2(\Psi, Q)$, we obtain that

$$\mathcal{S}_2(\Psi, Q) \ll (\log Q)^{16+\epsilon} \sum_{d>D} d^\epsilon \sum_{r \leq \frac{2Q}{d}} \phi(r) \ll (\log Q)^{16+\epsilon} \sum_{d>D} \left(\frac{Q}{d}\right)^2 d^\epsilon \ll \frac{Q^2 (\log Q)^{16+\epsilon}}{D^{1-\epsilon}}.$$

This term may be absorbed into the error term provided $D > \log Q$. □

6. TREATMENT OF $\mathcal{G}(\Psi, Q)$

6.1. The complementary divisor. Recall that

$$\mathcal{G}(\Psi, Q) = \frac{1}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m) \tau_4(n)}{\sqrt{mn}} \sum_{\substack{d,r \\ (dr,mn)=1 \\ r|m \pm n \\ d < D}} \mu(d) \phi(r) \Psi\left(\frac{dr}{Q}\right) V\left(m, n, \frac{dr}{(\log Q)^\alpha}\right).$$

We write $g = (m, n)$ and $m = gM$, $n = gN$, so that $(M, N) = 1$, and write $\phi(r) = \sum_{al=r} \mu(a)l$. The latter identity extracts the arithmetic information from $\phi(r)$ so that we

may eventually sum smoothly over the essential parts of the modulus. Then the sum over d and r is

$$\sum_{\substack{d,a,l \\ (d,mn)=1 \\ (al,g)=1 \\ al|M \pm N \\ d < D}} \mu(d)\mu(a)l\Psi\left(\frac{dal}{Q}\right) V\left(gM, gN, \frac{dal}{(\log Q)^\alpha}\right).$$

In the above, we have used that $(r, mn) = 1$ if and only if $(r, g) = 1$ since $r|m \pm n$. We let $|M \pm N| = alh$. We want to replace the condition modulo $r = al$ with a condition modulo h , which will be small when r is large. Thus we replace l with $\frac{|M \pm N|}{ah}$. To do so, we express the condition $(l, g) = 1$ by $\sum_{b|(l,g)} \mu(b)$. Writing $l = bk$, the sum becomes

$$\sum_{\substack{d < D \\ (d,gMN)=1}} \mu(d) \sum_{\substack{(a,g)=1}} \mu(a) \sum_{b|g} \mu(b) \sum_{\substack{k \geq 1 \\ |M-N|=abkh}} bk\Psi\left(\frac{dabk}{Q}\right) V\left(gM, gN, \frac{dabk}{(\log Q)^\alpha}\right).$$

We substitute $k = \frac{|M \pm N|}{abh}$ to get

$$(14) \quad Q \sum_{d < D} \sum_{(a,g)=1} \sum_{b|g} \sum_{\substack{h > 0 \\ M \equiv \mp N \pmod{abh}}} \frac{\mu(d)\mu(a)\mu(b)}{ad} \times \left(\frac{d|M \pm N|}{Qh}\right) \Psi\left(\frac{d|M \pm N|}{Qh}\right) V\left(gM, gN; \frac{d|M \pm N|}{hQ(\log Q)^\alpha}\right).$$

For non-negative real numbers u, x, y and for each choice of sign, we define

$$\mathcal{W}^\pm(x, y; u) = u|x \pm y| \Psi(u|x \pm y|) V(x, y; u|x \pm y|).$$

Since

$$V(m, n; \mu) = \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} W\left(mn \left(\frac{\pi}{\mu}\right)^4, t\right) dt,$$

we have

$$(15) \quad V(cm, cn; \sqrt{c}\mu) = \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} W\left(mn \left(\frac{\pi}{\mu}\right)^4, t\right) dt = V(m, n; \mu).$$

Thus (14) becomes

$$Q \sum_{d < D} \sum_{(a,g)=1} \sum_{b|g} \sum_{\substack{h > 0 \\ M \equiv \mp N \pmod{abh}}} \frac{\mu(d)\mu(a)\mu(b)}{ad} \mathcal{W}^\pm\left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}}\right).$$

Note that $(MN, abh) = 1$. We express the condition $M \equiv \mp N \pmod{abh}$ using characters $\chi \pmod{abh}$. We then separate the principal character contribution, which is the main term, and the non-principal characters which contribute to an acceptable error term. Specifically,

$$\mathcal{G}(\Psi, Q) = \mathcal{M}\mathcal{G}(\Psi, Q) + \mathcal{E}\mathcal{G}(\Psi, Q),$$

where

(16)

$$\mathcal{MG}(\Psi, Q) = \frac{Q}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{(a, g)=1 \\ b|g}} \sum_{\substack{h > 0 \\ (abh, MN)=1}} \sum_{\substack{h > 0 \\ (abh, MN)=1}} \frac{\mu(d)\mu(a)\mu(b)}{ad\phi(abh)} \mathcal{W}^{\pm} \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right),$$

and

(17)

$$\mathcal{EG}(\Psi, Q) = \frac{Q}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{(a, g)=1 \\ b|g}} \sum_{\substack{h > 0 \\ (abh, MN)=1}} \sum_{\substack{h > 0 \\ (abh, MN)=1}} \frac{\mu(d)\mu(a)\mu(b)}{ad\phi(abh)} \\ \times \sum_{\substack{\chi \pmod{abh} \\ \chi \neq \chi_0}} \chi(M)\bar{\chi}(\mp N) \mathcal{W}^{\pm} \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right).$$

6.2. Mellin transforms of \mathcal{W}^{\pm} . A simple heuristic estimate of \mathcal{MG} tells us that it is, up to a factor of a power of $\log Q$, of size Q^2 (for this estimate, note that h is the essential part of the modulus abh). Since we expect cancellation in the sum when χ is non-principal, we expect \mathcal{EG} to be small.

To evaluate $\mathcal{MG}(\Psi, Q)$ and $\mathcal{EG}(\Psi, Q)$ more precisely, we write \mathcal{W}^{\pm} in terms of its Mellin transforms. There are three different types that we shall consider. They come from taking Mellin transforms in the variable u for when we need to sum over the modulus h , the variables x, y for when we need to sum over M and N , and all three variables for when we need to sum over M , N , and h . (In the description above, we have neglected to mention the conceptually less important sums over d, a, b and g .)

We collect the properties of the various Mellin transforms in the following three lemmas. The proofs of the lemmas are the same as the ones in Section 7 of [5], but using the bounds of Lemma 2 instead.

Lemma 7. *Given positive real numbers x and y , define*

$$\widetilde{\mathcal{W}}_1^{\pm}(x, y; z) = \int_0^{\infty} \mathcal{W}^{\pm}(x, y; u) u^z \frac{du}{u}.$$

Then the functions $\widetilde{\mathcal{W}}_1^{\pm}(x, y; z)$ are analytic for all $z \in \mathbb{C}$. We have the Mellin inversion formula

$$(18) \quad \mathcal{W}^{\pm}(x, y; u) = \frac{1}{2\pi i} \int_{(c)} \widetilde{\mathcal{W}}_1^{\pm}(x, y; z) u^{-z} dz,$$

where the integral is taken over the line $\operatorname{Re}(z) = c$ for any real number c . The Mellin transforms $\widetilde{\mathcal{W}}_1^{\pm}(x, y; z)$ satisfy for any non-negative integer ν

$$|\widetilde{\mathcal{W}}_1^{\pm}(x, y; z)| \ll_{\nu} |x \pm y|^{-\operatorname{Re} z} \prod_{j=1}^{\nu} |z + j|^{-1} \exp(-c_1 \max(x, y)^{1/4})$$

for some absolute constant c_1 .

Lemma 8. *Given a positive real number u , we define*

$$\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u) = \int_0^\infty \int_0^\infty \mathcal{W}^\pm(x, y; u) x^{s_1} y^{s_2} \frac{dx}{x} \frac{dy}{y}.$$

Then the functions $\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u)$ are analytic in the region $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > 0$. We have the Mellin inversion formula

$$\mathcal{W}^\pm(x, y; u) = \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u) x^{-s_1} y^{-s_2} ds_1 ds_2,$$

where c_1, c_2 are positive. The Mellin transforms $\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u)$ satisfy, for any $k \geq 1$

$$|\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u)| \ll \frac{(1+u)^{k-1}}{\max(|s_1|, |s_2|)^k} \exp(-c_1 u^{-1/4}).$$

Lemma 9. *We define*

$$\widetilde{\mathcal{W}}_3^\pm(s_1, s_2; z) = \int_0^\infty \int_0^\infty \mathcal{W}^\pm(x, y; u) u^z x^{s_1} y^{s_2} \frac{du}{u} \frac{dx}{x} \frac{dy}{y},$$

and

$$\widetilde{\mathcal{W}}_3(s_1, s_2; z) = \widetilde{\mathcal{W}}_3^+(s_1, s_2; z) + \widetilde{\mathcal{W}}_3^-(s_1, s_2; z).$$

Let $\omega = \frac{s_1+s_2-z}{2}$ and $\xi = \frac{s_1-s_2+z}{2}$. For $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > 0$, and $|\operatorname{Re}(s_1-s_2)| < \operatorname{Re}(z) < 1$ we have

$$(19) \quad \widetilde{\mathcal{W}}_3(s_1, s_2; z) = \frac{\widetilde{\Psi}(1+4\omega+z)}{2\omega\pi^{4\omega}} \int_{-\infty}^\infty \mathcal{H}(\xi-it, z) G\left(\frac{1}{2}+\omega, t\right) dt,$$

where $\widetilde{\Psi}$ is defined in (13), and

$$\mathcal{H}(u, v) = \pi^{1/2} \frac{\Gamma\left(\frac{u}{2}\right) \Gamma\left(\frac{1-v}{2}\right) \Gamma\left(\frac{v-u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right) \Gamma\left(\frac{v}{2}\right) \Gamma\left(\frac{1-v+u}{2}\right)}.$$

Moreover we have the Mellin inversion formula

$$\mathcal{W}^\pm(x, y; u) = \frac{1}{(2\pi i)^3} \int_z \int_{s_1} \int_{s_2} \widetilde{\mathcal{W}}_2^\pm(s_1, s_2; z) u^{-z} x^{-s_1} y^{-s_2} ds_2 ds_1 dz$$

where all of the paths are taken to be the vertical lines with increasing imaginary parts and real parts satisfying the constraints given above, and the integrals over s_1 and s_2 are to be interpreted as being over $|\operatorname{Im}(s_1)| \leq T_1$ and $|\operatorname{Im}(s_2)| \leq T_s$ and letting T_1, T_2 tend to infinity. Finally the Mellin transform $\widetilde{\mathcal{W}}_3(s_1, s_2; z)$ satisfies the bound

$$(20) \quad |\widetilde{\mathcal{W}}_3(s_1, s_2; z)| \ll (1+|z|)^{-A} (1+|\omega|)^{-A} (1+|\xi|)^{\operatorname{Re}(z)-1}.$$

7. BOUNDING THE ERROR TERM $\mathcal{EG}(\Psi, Q)$

In this section, we write β for the absolute positive constant, which may stand for different values from line to line. First we show that we can restrict the sum over a to $a \leq 2Q$. Since $M \neq N$, if $a > 2Q$ and $M \equiv \mp N \pmod{abh}$ then

$$\frac{Qd}{gh(\log Q)^{2\alpha}} \frac{|gM \pm gN|(\log Q)^{2\alpha}}{Q^2} \geq \frac{dab}{Q} \geq 2,$$

so $\mathcal{W}^\pm\left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}}\right) = 0$. Therefore

$$\mathcal{EG}(\Psi, Q) = \mathcal{EG}_1(\Psi, Q) - \mathcal{EG}_2(\Psi, Q),$$

where

$$\begin{aligned} \mathcal{EG}_1(\Psi, Q) &= \frac{Q}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{(a,g)=1 \\ a \leq 2Q}} \sum_{b|g} \sum_{\substack{h>0 \\ (abh, MN)=1}} \frac{\mu(d)\mu(a)\mu(b)}{ad\phi(abh)} \\ &\quad \times \sum_{\substack{\chi \pmod{abh} \\ \chi \neq \chi_0}} \chi(M)\overline{\chi}(\mp N) \mathcal{W}^{\pm} \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{EG}_2(\Psi, Q) &= \frac{Q}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{(a,g)=1 \\ a > 2Q}} \sum_{b|g} \sum_{\substack{h>0 \\ (abh, MN)=1}} \frac{\mu(d)\mu(a)\mu(b)}{ad\phi(abh)} \\ &\quad \times \mathcal{W}^{\pm} \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right). \end{aligned}$$

Since Ψ is supported in $[1, 2]$, $\frac{|M \pm N|d}{Qh} \in [1, 2]$. From Lemma 2, we have that

$$\mathcal{W}^{\pm} \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right) \ll \exp \left(-c \left(\frac{\max(m, n)(\log Q)^{2\alpha}}{Q^2} \right) \right),$$

so $\mathcal{W}^{\pm} \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right)$ is small unless $m, n \leq \frac{Q^2}{(\log Q)^{\alpha}}$. Moreover, $\phi(abh) \gg abh \log \log(abh)$, and $\sum_{m \leq x} \frac{\tau_4(m)}{\sqrt{m}} \ll \sqrt{x}(\log x)^3$. Therefore

$$\begin{aligned} \mathcal{EG}_2(\Psi, Q) &\ll Q(\log Q)^{\epsilon} \sum_{\substack{m,n \leq \frac{Q^2}{(\log Q)^{\alpha}}}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{d \leq D} \sum_{a > 2Q} \sum_{b \leq Q^2} \sum_{h \leq Q/d} \frac{1}{a^2 b d h} \\ &\ll Q(\log Q)^{\epsilon} \sum_{\substack{m,n \leq \frac{Q^2}{(\log Q)^{\alpha}}}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{a > 2Q} \frac{1}{a^2} \ll \frac{Q^2}{(\log Q)^{\alpha-1}}, \end{aligned}$$

which is an acceptable error term. Now we will bound $\mathcal{EG}_1(\Psi, Q)$. By Lemma 8, we can write $\mathcal{EG}_1(\Psi, Q)$ as

$$\begin{aligned} &\frac{Q}{2} \sum_{\substack{a \leq 2Q \\ b, h > 0}} \sum_{\substack{\chi \pmod{abh} \\ \chi \neq \chi_0}} \sum_{\substack{g \\ b|g, (a,g)=1}} \sum_{\substack{d \leq D \\ (d, g)=1}} \frac{\mu(a)\mu(b)\mu(d)}{adg\phi(abh)} \frac{1}{(2\pi i)^2} \int_{(1/2+\epsilon)} \int_{(1/2+\epsilon)} \widetilde{\mathcal{W}}_2^{\pm} \left(s_1, s_2; \frac{Qd}{gh(\log Q)^{2\alpha}} \right) \\ &\quad \times \left(\frac{Q^2}{g(\log Q)^{2\alpha}} \right)^{s_1+s_2} \sum_{\substack{M, N=1 \\ M \neq N, (M, N)=1 \\ (MN, d)=1}}^{\infty} \frac{\tau_4(gM)\tau_4(gN)}{M^{1/2+s_1} N^{1/2+s_2}} \chi(M)\overline{\chi}(\mp N) ds_1 ds_2. \end{aligned}$$

We write the inner sum over M and N as

$$\chi(\mp 1) \left(L^4(1/2 + s_1, \chi) L^4(1/2 + s_2, \overline{\chi}) \lambda(g, s_1, s_2, \chi) \theta(g, s_1, s_2, \chi; d) - \tau_4^2(g) \right),$$

where $\lambda(g, s_1, s_2, \chi)$ is holomorphic when $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > \epsilon$. If $\operatorname{Re}(s_1), \operatorname{Re}(s_2) = \frac{100}{\log Q}$, $\lambda(g, s_1, s_2, \chi)$ is

$$\prod_p \left(1 + \sum_{k, l \geq 1} \frac{\chi(p^k) \tau_4(p^k)}{p^{k(1/2+s_1)}} \frac{\overline{\chi(p^l)} \tau_4(p^l)}{p^{l(1/2+s_2)}} \left(1 + \sum_{k \geq 1} \frac{\chi(p^k) \tau_4(p^k)}{p^{k(1/2+s_1)}} + \sum_{k \geq 1} \frac{\overline{\chi(p^k)} \tau_4(p^k)}{p^{k(1/2+s_2)}} \right)^{-1} \right)^{-1} \\ \ll (\log Q)^{16}.$$

Also, for these values of s_1, s_2 , $\theta(g, s_1, s_2, \chi; d)$ is

$$\tau_4(g) \prod_{p|dg} \left(1 + \sum_{k \geq 1} \frac{\chi(p^k) \tau_4(p^k)}{p^{k(1/2+s_1)}} + \sum_{k \geq 1} \frac{\overline{\chi(p^k)} \tau_4(p^k)}{p^{k(1/2+s_2)}} \right)^{-1} \prod_{p^r || g} \left(\sum_{k \geq 0} \frac{\tau_4(p^{r+k})}{p^{k(1/2+s_1)}} + \sum_{k \geq 1} \frac{\tau_4(p^{r+k})}{p^{k(1/2+s_2)}} \right) \\ \ll \tau^3(dg) \tau_4^2(g),$$

where $\tau(n)$ is the usual divisor function. Since χ is not principal, we can shift contours to $\operatorname{Re}(s_1) = \operatorname{Re}(s_2) = \frac{100}{\log Q}$ without passing any poles. There, $\mathcal{E}\mathcal{G}_1(\Psi, Q)$ is bounded by

$$\ll Q(\log Q)^{16} \sum_{\substack{a \leq 2Q \\ b, h > 0}} \sum_{\substack{\chi \pmod{abh} \\ \chi \neq \chi_0}} \sum_{\substack{g \\ b|g, (a, g)=1}} \sum_{\substack{d \leq D \\ (d, g)=1}} \frac{\tau^3(d) \tau^3(g) \tau_4(g)}{adg \phi(abh)} \\ (21) \\ \times \int_{(\frac{100}{\log Q})} \int_{(\frac{100}{\log Q})} (|L^4(1/2 + s_1, \chi) L^4(1/2 + s_2, \bar{\chi})| + \tau_4^2(g)) \left| \widetilde{\mathcal{W}}_2^\pm \left(s_1, s_2; \frac{Qd}{gh(\log Q)^{2\alpha}} \right) \right| ds_1 ds_2.$$

From Lemma 8, we have that for any $k \geq 1$ and $t = 1$ or 3 ,

$$\sum_{\substack{g \\ b|g}} \frac{\tau^3(g) \tau_4^t(g)}{g} \sum_{d \leq D} \frac{\tau^3(d)}{d} \left| \widetilde{W}_2 \left(s_1, s_2, \frac{Qd}{gh(\log Q)^{2\alpha}} \right) \right| \\ \ll \frac{\left(1 + \frac{QD}{bh(\log Q)^{2\alpha}} \right)^{k-1}}{\max(|s_1|, |s_2|)^k} (\log D)^6 \sum_{\substack{g \\ b|g}} \frac{\tau^3(g) \tau_4^t(g)}{g} \exp \left(-c \left(\frac{gh(\log Q)^{2\alpha}}{QD} \right)^{1/4} \right) \\ (22) \ll \frac{\left(1 + \frac{QD}{bh(\log Q)^{2\alpha}} \right)^{k-1}}{\max(|s_1|, |s_2|)^k} \frac{(\log Q)^\beta \tau^3(b) \tau_4^3(b)}{b} \exp \left(-c \left(\frac{bh(\log Q)^{2\alpha}}{QD} \right)^{1/4} \right).$$

Now examine dyadic intervals $S_1 \leq |s_1| < 2S_1$, $S_2 \leq |s_2| < 2S_2$, $A \leq a < 2A$, $B \leq b < 2B$, and $H \leq h < 2H$. We write $\ell = abh$. Note that $\phi(abh) \gg abh \log \log(abh)$, and $\tau^3(b) \tau_4^3(b) \leq \tau^3(\ell) \tau_4^3(\ell)$. Thus, the contribution of such a dyadic block is $Q(\log Q)^\beta$ times a quantity

$$\ll \frac{\left(1 + \frac{QD}{BH(\log Q)^{2\alpha}} \right)^{k-1} \log^\epsilon(ABH)}{A^2 B^2 H \max(S_1, S_2)^k} \exp \left(-c \left(\frac{BH(\log Q)^{2\alpha}}{QD} \right)^{1/4} \right) \sum_{ABH \leq \ell < 8ABH} \tau^6(\ell) \tau_4^3(\ell) \\ (23)$$

$$\times \sum_{\substack{\chi \pmod{\ell} \\ \chi \neq \chi_0}} \int_{S_1 \leq |s_1| < 2S_1} \int_{S_2 \leq |s_2| < 2S_2} (1 + |L^4(1/2 + s_1, \chi) L^4(1/2 + s_2, \bar{\chi})|) ds_1 ds_2$$

Let $S = \max(S_1, S_2)$. By Proposition 1,

$$\begin{aligned}
& \sum_{\substack{\chi \pmod{\ell} \\ \chi \neq \chi_0}} \int_{S_1 \leq |s_1| < 2S_1} \int_{S_2 \leq |s_2| < 2S_2} (1 + |L^4(1/2 + s_1, \chi)L^4(1/2 + s_2, \bar{\chi})|) \, ds_1 \, ds_2 \\
& \leq \sum_{\substack{\chi \pmod{\ell} \\ \chi \neq \chi_0}} \int_{S_1 \leq |s_1| < 2S_1} \int_{S_2 \leq |s_2| < 2S_2} (1 + |L(1/2 + s_1, \chi)|^8 + |L(1/2 + s_2, \bar{\chi})|^8) \, ds_1 \, ds_2 \\
(24) \quad & \ll lS_1S_2 \log^{16+\epsilon}(l(S+1)).
\end{aligned}$$

We let \sum^d denote a dyadic sum. By (22), (23), and (24) above, (21) is bounded by $Q \log^\beta Q$ times

$$\begin{aligned}
& \sum_{A,B,H,S}^d \frac{\left(1 + \frac{QD}{BH(\log Q)^{2\alpha}}\right)^{k-1} \log^{16+\epsilon}(ABH(S+1))}{ABS^{k-2}} \exp\left(-c \left(\frac{BH(\log Q)^{2\alpha}}{QD}\right)^{1/4}\right) \times \\
& \quad \times \sum_{ABH \leq \ell < 8ABH} \tau^6(\ell) \tau_4^3(\ell) \\
& \ll \sum_{B,H,S}^d \frac{H \left(1 + \frac{QD}{BH(\log Q)^{2\alpha}}\right)^{k-1} \log^\beta((S+1)QBH)}{S^{k-2}} \exp\left(-c \left(\frac{BH(\log Q)^{2\alpha}}{QD}\right)^{1/4}\right),
\end{aligned}$$

where we have removed the sum over A by using that $A \leq 2Q$. We take $k = 1$ if $S \leq 1 + \frac{QD}{BH(\log Q)^{2\alpha}}$ and $k = 4$ otherwise. The contribution from $S \leq 1 + \frac{QD}{BH(\log Q)^{2\alpha}}$ is

$$\begin{aligned}
& \ll \sum_{B,H}^d H \left(1 + \frac{QD}{BH(\log Q)^{2\alpha}}\right) \log^\beta(QBH) \exp\left(-c \left(\frac{BH(\log Q)^{2\alpha}}{QD}\right)^{1/4}\right) \\
& \ll \sum_L^d L \left(1 + \frac{QD}{L(\log Q)^{2\alpha}}\right) \log^\beta(QL) \exp\left(-c \left(\frac{L(\log Q)^{2\alpha}}{QD}\right)^{1/4}\right) \\
& \ll \frac{QD}{(\log Q)^{2\alpha-\beta}}
\end{aligned}$$

When $S > 1 + \frac{QD}{BH(\log Q)^{2\alpha}}$, picking $k = 4$ and excuting the dyadic sum over S gives that the contribution is also bounded by

$$\sum_{B,H}^d H \left(1 + \frac{QD}{BH(\log Q)^{2\alpha}}\right) \log^\beta(QBH) \exp\left(-c \left(\frac{BH(\log Q)^{2\alpha}}{QD}\right)^{1/4}\right) \ll \frac{QD}{(\log Q)^{2\alpha-\beta}}.$$

To summarize our work, we have obtained that $\mathcal{EG}_1(\Phi, Q) \ll \frac{Q^2 D}{(\log Q)^{2\alpha-\beta}}$. If $D = (\log Q)^\delta$ for a fixed number δ independent of α , then

$$\mathcal{EG}_1(\Phi, Q) \ll \frac{Q^2}{\log Q},$$

is an acceptable error term, provided we pick α large enough.

8. EVALUATING $\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q)$

We recall that $\mathcal{MG}(\Psi, Q)$ is defined in (16). We first simplify the sum over a, b in $\mathcal{MG}(\Psi, Q)$ by letting $r = ab$ so that

$$\mathcal{A}(h, g, MN) = \sum_{(a, gMN)=1} \sum_{\substack{b|g \\ (b, MN)}} \frac{\mu(a)\mu(b)}{a\phi(abh)} = \sum_{(r, MN)=1} \frac{\mu(r)(r, g)}{r\phi(rh)}.$$

Next we will evaluate sum over h , which is given by

$$\sum_{(h, MN)=1} \mathcal{A}(h, g, MN) \mathcal{W}_1^\pm \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; \frac{Qd}{gh(\log Q)^{2\alpha}} \right).$$

Using Mellin transform of $\widetilde{\mathcal{W}}_1^\pm$ given in Lemma 7 with $c = -\epsilon < 0$, we obtain that the above sum is

$$(25) \quad \sum_{\substack{h=1 \\ (h, MN)=1}}^{\infty} \mathcal{A}(h, g, MN) \frac{1}{2\pi i} \int_{(-\epsilon)} \widetilde{\mathcal{W}}_1^\pm \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; z \right) \left(\frac{Qd}{gh(\log Q)^{2\alpha}} \right)^{-z} dz.$$

We can interchange the sum and the integral since the sum over h is absolutely convergent for $\Re(z) < 0$. Writing out the Euler product, we obtain that

$$\sum_{\substack{h=1 \\ (h, MN)=1}}^{\infty} \frac{\mathcal{A}(h, g; MN)}{h^s} = \zeta(s+1) \mathcal{F}(s, g, MN),$$

where

$$(26) \quad \mathcal{F}(s, g, MN) = \phi(MN, s+1) \prod_{p \nmid gMN} \left(1 - \frac{1}{p(p-1)} + \frac{1}{p^{1+s}(p-1)} \right) \prod_{\substack{p|g \\ p \nmid MN}} \left(1 - \frac{1}{p^{1+s}} - \frac{1}{p-1} \left(1 - \frac{1}{p^s} \right) \right),$$

and

$$\phi(r, s) = \prod_{p|r} \left(1 - \frac{1}{p^s} \right).$$

Therefore (25) is

$$(27) \quad \begin{aligned} & \frac{1}{2\pi i} \int_{(-\epsilon)} \widetilde{\mathcal{W}}_1^\pm \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; z \right) \zeta(1-z) \mathcal{F}(-z, g, MN) \left(\frac{Qd}{g(\log Q)^{2\alpha}} \right)^{-z} dz \\ &= -(\text{Residue at } z=0) + \\ &+ \frac{1}{2\pi i} \int_{(\epsilon)} \widetilde{\mathcal{W}}_1^\pm \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; z \right) \zeta(1-z) \mathcal{F}(-z, g, MN) \left(\frac{Qd}{g(\log Q)^{2\alpha}} \right)^{-z} dz. \end{aligned}$$

By the definition of $\mathcal{F}(0, g, MN)$, we have that $-(\text{Residue at } z=0)$ is

$$\begin{aligned} & \widetilde{\mathcal{W}}_1^\pm \left(\frac{gM(\log Q)^{2\alpha}}{Q^2}, \frac{gN(\log Q)^{2\alpha}}{Q^2}; 0 \right) \zeta(1-z) \mathcal{F}(0, g, MN) \\ &= \frac{\phi(mn)}{mn} \int_0^\infty \Psi(u) V \left(\frac{m(\log Q)^{2\alpha}}{Q^2}, \frac{n(\log Q)^{2\alpha}}{Q^2}; u \right) du. \end{aligned}$$

Therefore the contribution of the residue term at $z = 0$ is

$$(28) \quad Q \sum_{\substack{m,n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d,mn)=1}} \frac{\mu(d)}{d} \frac{\phi(mn)}{mn} \int_0^\infty \Psi(u) V\left(\frac{m(\log Q)^{2\alpha}}{Q^2}, \frac{n(\log Q)^{2\alpha}}{Q^2}; u\right) du.$$

We will show that the sum of the quantity in Equation (28) with $\mathcal{MS}(\Psi, Q)$ gives an acceptable error term. Recall the definition of $\mathcal{MS}(\Psi, Q)$ is

$$- \sum_{\substack{m,n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d,mn)=1}} \mu(d) \sum_{(r,mn)=1} \Psi\left(\frac{dr}{Q}\right) V\left(m, n; \frac{dr}{(\log Q)^\alpha}\right).$$

By elementary arguments, we have

$$\sum_{\substack{r \leq x \\ (r,mn)=1}} 1 = \frac{\phi(mn)}{mn} x + O(\tau(mn)),$$

where τ is the divisor function. By partial summation, the main contribution from $\mathcal{MS}(\Psi, Q)$ is

$$- Q \sum_{\substack{m,n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d,mn)=1}} \frac{\mu(d)}{d} \frac{\phi(mn)}{mn} \int_0^\infty \Psi(u) V\left(m, n; \frac{uQ}{(\log Q)^\alpha}\right) du = - \text{Equation (28)}$$

by relation (15). Since V is small unless $\max(m, n) \leq \frac{Q^2}{(\log Q)^{\alpha/2}}$, we then obtain that $\mathcal{MS}(\Psi, Q) + \text{Equation (28)}$ is bounded by $\frac{DQ^2(\log Q)^\beta}{(\log Q)^{\alpha/2}}$, where β does not depend on α . Therefore the main term of $\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q)$ is

$$\begin{aligned} & \frac{Q}{2} \sum_{\substack{m,n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{2\pi i} \int_{(\epsilon)} \widetilde{\mathcal{W}}_1^\pm\left(\frac{m(\log Q)^{2\alpha}}{Q^2}, \frac{n(\log Q)^{2\alpha}}{Q^2}; z\right) \\ & \quad \times \zeta(1-z) \mathcal{F}(-z, g, MN) \left(\frac{Q}{g(\log Q)^{2\alpha}}\right)^{-z} \sum_{\substack{d \leq D \\ (d,gMN)=1}} \frac{\mu(d)}{d^{1+z}} dz. \end{aligned}$$

Now we shift the contour to $\text{Re}(z) = 1 - \frac{1}{\log Q}$. We extend the sum over d to all positive integers. By Lemma 7, $\widetilde{\mathcal{W}}_1^\pm$ is small unless m, n and $g \leq Q^2$. By Equation (26), the error term in extending is

$$\begin{aligned} & \ll Q \sum_{\substack{m,n=1 \\ m \neq n}} \frac{\tau_4(m)\tau(m)\tau_4(n)\tau(n)\tau^2(g)}{\sqrt{mn}} \int_{(1-\frac{1}{\log Q})} \left| \frac{(m \pm n)(\log Q)^{2\alpha}}{Q^2} \right|^{-1+1/\log Q} |z|^{-10} \\ & \quad \times \exp\left(-c \left(\frac{\max(m, n)(\log Q)^{2\alpha}}{Q^2}\right)^{1/4}\right) \left(\frac{Q}{g(\log Q)^{2\alpha}}\right)^{-1+1/\log Q} \frac{1}{D} |dz| \end{aligned}$$

(29)

$$\ll \frac{Q^2}{D} \sum_{g \leq Q^2} \frac{\tau_4^2(g)\tau^2(g)}{g} \sum_{\substack{m,n \leq Q^2 \\ m \neq n}} \frac{\tau_4(m)\tau(m)\tau_4(n)\tau(n)}{\sqrt{mn}} \frac{1}{|m \pm n|}$$

Since $m + n \geq 2\sqrt{mn}$, we have

$$\sum_{\substack{m, n \leq Q^2 \\ m \neq n}} \frac{\tau_4(m)\tau(m)\tau_4(n)\tau(n)}{\sqrt{mn}} \frac{1}{|m+n|} \ll \sum_{m, n \leq Q^2} \frac{\tau_4(m)\tau(m)\tau_4(n)\tau(n)}{mn} \ll (\log Q)^{16}.$$

For the other term, we may assume by symmetry that $m < n$ and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{\substack{m, n \leq Q^2 \\ m < n}} \frac{\tau_4(m)\tau(m)\tau_4(n)\tau(n)}{\sqrt{mn}|m-n|} &\ll \sum_{m \leq Q^2} \frac{\tau_4(m)\tau(m)}{\sqrt{m}} \sum_{\ell \leq Q^2} \frac{\tau_4(m+\ell)\tau(m+\ell)}{\ell\sqrt{m+\ell}} \\ &\ll \sum_{\ell \leq Q^2} \frac{1}{\ell} \left(\sum_{m \leq Q^2} \frac{\tau_4^2(m)\tau^2(m)}{m} \right)^{1/2} \left(\sum_{m \leq Q^2} \frac{\tau_4^2(m+\ell)\tau^2(m+\ell)}{m+\ell} \right)^{1/2} \\ &\ll (\log Q)^{65}. \end{aligned}$$

Therefore (29) is bounded above by $\frac{Q^2(\log Q)^{129}}{D}$.

We now move the contour back to $\text{Re}(z) = \epsilon$ and reinsert the terms $m = n$ with a negligible error of $O(Q^{1+\epsilon})$. Hence up to the error term of $\frac{DQ^2(\log Q)^\beta}{(\log Q)^{\alpha/2}} + \frac{Q^2(\log Q)^{129}}{D}$, $\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q)$ is

$$(30) \quad \frac{Q}{2} \sum_{m, n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{2\pi i} \int_{(\epsilon)} \widetilde{\mathcal{W}}_1^{\pm} \left(\frac{m(\log Q)^{2\alpha}}{Q^2}, \frac{n(\log Q)^{2\alpha}}{Q^2}; z \right) \\ \times \frac{\zeta(1-z)\mathcal{F}(-z, g, MN)}{\zeta(1+z)\phi(gMN, 1+z)} \left(\frac{Q}{g(\log Q)^{2\alpha}} \right)^{-z} dz.$$

Choosing $D = (\log Q)^{130}$ and α large enough, we obtain that the error term is $O\left(\frac{Q^2}{\log Q}\right)$.

9. THE OFF-DIAGONAL CONTRIBUTION

In this section, we will evaluate (30). First we use Mellin transform in Lemma 9 and obtain that (30) is

$$(31) \quad \frac{Q}{2} \frac{1}{(2\pi i)^3} \int_{(\epsilon)} \int_{(1/2+\epsilon)} \int_{(1/2+\epsilon)} \widetilde{\mathcal{W}}_3(s_1, s_2; z) \frac{\zeta(1-z)}{\zeta(1+z)} \frac{Q^{2s_1+2s_2-z}}{(\log Q)^{2\alpha(s_1+s_2-z)}} \mathcal{B}(s_1, s_2; z) ds_2 ds_1 dz,$$

where

$$\mathcal{J}(s_1, s_2; z) = \sum_{m, n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{m^{1/2+s_1}n^{1/2+s_2}} \frac{g^z \mathcal{F}(-z, g, MN)}{\phi(gMN, 1+z)}.$$

The coefficients above are multiplicative, so we can write $\mathcal{J}(s_1, s_2; z) = \prod_p \mathcal{J}_p(s_1, s_2; z)$, where

$$(32) \quad \mathcal{J}_p(s_1, s_2; z) = 1 + \frac{p^z - 1}{p(p-1)} + \sum_{\substack{a, b \geq 0 \\ \max(a, b) \geq 1}} \frac{\tau_4(p^a)\tau_4(p^b)}{p^{a(1/2+s_1)}p^{b(1/2+s_2)}} p^{z \min(a, b)} \frac{1 - \frac{1}{p^{1-z}}}{1 - \frac{1}{p^{1+z}}} \\ + \frac{1}{(p-1)} \frac{p^z - 1}{1 - \frac{1}{p^{1+z}}} \sum_{k=1}^{\infty} \frac{\tau_4^2(p^k)}{p^{k(1+s_1+s_2-z)}}.$$

Thus $\mathcal{J}(s_1, s_2; z)$ as

$$\zeta(2-z) \frac{\zeta^4\left(\frac{1}{2}+s_1\right) \zeta^4\left(\frac{1}{2}+s_2\right)}{\zeta^4\left(\frac{3}{2}+s_1-z\right) \zeta^4\left(\frac{3}{2}+s_2-z\right)} \zeta^{16}(1+s_1+s_2-z) \mathcal{K}(s_1, s_2; z),$$

where $\mathcal{K}(s_1, s_2; z) = \prod_p \mathcal{K}_p(s_1, s_2; z)$ is absolute convergent in $\operatorname{Re}(s_1) > 0, \operatorname{Re}(s_2) > 0$ and $\operatorname{Re}(s_1 + s_2) > \operatorname{Re}(z) - 1/2$.

Before moving the line of integration, we first truncate the integrals in s_1 and s_2 at a height $T = Q^2$. Using the bound for $\widetilde{\mathcal{W}}_3$ from Equation (20), we obtain that the error term from the truncation is $O(Q^{1+\epsilon})$. Now we move the line of integration in s_1 and s_2 to $\operatorname{Re}(s_1) = \operatorname{Re}(s_2) = \frac{2}{\log Q}$, encountering poles of order four at $s_1 = 1/2$ and $s_2 = 1/2$.

Using the Lindelof hypothesis and the bound (20) for $\widetilde{\mathcal{W}}_3$, the remaining integral is also $O(Q^{1+\epsilon})$, so we need only study the contribution from the residues.

Recall that

$$\widetilde{\mathcal{W}}_3(s_1, s_2; z) = \frac{\widetilde{\Psi}(1+2s_1+2s_2-z)}{(s_1+s_2-z)\pi^{2s_1+2s_2-2z}} \int_{-\infty}^{\infty} \mathcal{H}\left(\frac{s_1-s_2+z}{2}-it, z\right) G\left(\frac{1}{2}+\frac{s_1+s_2-z}{2}, t\right) dt.$$

Thus the residue from $s_1 = s_2 = 1/2$ is

$$(33) \quad \begin{aligned} & \frac{Q}{2} \frac{1}{2\pi i} \int_{(\epsilon)} \frac{\widetilde{\Psi}(3-z)}{\pi^{2-2z}} \left(\int_{-\infty}^{\infty} \mathcal{H}\left(\frac{z}{2}-it, z\right) G\left(\frac{1}{2}+\frac{1-z}{2}, t\right) dt \right) \frac{\zeta(1-z)\zeta(2-z)\mathcal{K}\left(\frac{1}{2}, \frac{1}{2}; z\right)}{\zeta(1+z)\zeta^8(2-z)} \\ & \times \left(\operatorname{Res}_{s_1=s_2=1/2} \frac{\zeta^4\left(\frac{1}{2}+s_1\right) \zeta^4\left(\frac{1}{2}+s_2\right) \zeta^{16}(1+s_1+s_2-z)}{(s_1+s_2-z)} \frac{Q^{2s_1+2s_2-z}}{(\log Q)^{2\alpha(s_1+s_2-z)}} \right) dz. \end{aligned}$$

Note this contributes a factor of $\log^6 Q$ to the leading term. From the definition of \mathcal{H} in Lemma 9, we have

$$\mathcal{H}\left(\frac{z}{2}-it, z\right) = \pi^{1/2} \frac{\Gamma\left(\frac{z}{4}-\frac{it}{2}\right) \Gamma\left(\frac{1-z}{2}\right) \Gamma\left(\frac{z}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{z}{4}+\frac{it}{2}\right) \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{1}{2}-\frac{z}{4}-\frac{it}{2}\right)}.$$

We now move the integration in z to $\operatorname{Re}(z) = \frac{3}{2} - \frac{1}{\log Q}$. The remaining integral gives an acceptable error term, and we encounter a pole of order 11 at $z = 1$ (there is a simple pole from $\mathcal{H}\left(\frac{z}{2}-it, z\right)$, a zero of order 7 from $\frac{1}{\zeta^7(2-z)}$ and a pole of order 17 from the expression $\operatorname{Res}_{s_1=s_2=1/2}$). Therefore the leading term of the residue at $z = 1$ has order $Q^2(\log Q)^{16}$. By Maple (or a tedious calculation by hand), we obtain that the leading term is

$$\begin{aligned} & \frac{13381}{2615348736000} Q^2 (\log Q)^{16} \zeta(0) \frac{\widetilde{\Psi}(2)}{2} \frac{\mathcal{K}\left(\frac{1}{2}, \frac{1}{2}; 1\right)}{\zeta(2)} \int_{-\infty}^{\infty} G\left(\frac{1}{2}, t\right) dt \\ & = \frac{-53524}{16!} Q^2 (\log Q)^{16} \frac{\widetilde{\Psi}(2)}{2} \frac{\mathcal{K}\left(\frac{1}{2}, \frac{1}{2}; 1\right)}{\zeta(2)} \int_{-\infty}^{\infty} G\left(\frac{1}{2}, t\right) dt. \end{aligned}$$

In conclusion, we have that

$$\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q) = \frac{-53524}{16!} Q^2 (\log Q)^{16} \frac{\widetilde{\Psi}(2)}{2} \frac{\mathcal{K}\left(\frac{1}{2}, \frac{1}{2}; 1\right)}{\zeta(2)} \int_{-\infty}^{\infty} G\left(\frac{1}{2}, t\right) dt + O(Q^2 (\log Q)^{15}).$$

10. CONCLUSION OF THE PROOF OF THEOREM 1

By Sections 5 to 9, we obtain that

$$\mathcal{S}(\Psi, Q) + \mathcal{G}(\Psi, Q) = \frac{-53524}{16!} Q^2 (\log Q)^{16} \frac{\tilde{\Psi}(2)}{2} \frac{\mathcal{K}(\frac{1}{2}, \frac{1}{2}; 1)}{\zeta(2)} \int_{-\infty}^{\infty} G\left(\frac{1}{2}, t\right) dt + O(Q^2 (\log Q)^{15+\epsilon}).$$

Moreover, by Equation (12) in Section 4.1, we have that $\mathcal{D}(\Psi, Q)$, up to error term of $O(Q^2 (\log Q)^{15+\epsilon})$, is

$$2^{16} Q^2 \frac{(\log Q)^{16}}{16!} \frac{\tilde{\Psi}(2)}{2} A(1/2) \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{B_p(1/2)} \left(\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right)\right) \int_{-\infty}^{\infty} G(1/2, t) dt$$

To derive an expression for $\mathcal{S}(\Psi, Q) + \mathcal{G}(\Psi, Q) + \mathcal{D}(\Psi, Q)$, we first show that

$$\frac{\mathcal{K}(\frac{1}{2}, \frac{1}{2}; 1)}{\zeta(2)} = A(1/2) \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{B_p(1/2)} \left(\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right)\right).$$

It suffices to check that

$$(34) \quad \mathcal{K}_p\left(\frac{1}{2}, \frac{1}{2}; 1\right) \left(1 - \frac{1}{p^2}\right) = A_p(1/2) \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{B_p(1/2)} \left(\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right)\right).$$

By the definition of A_p , B_p , and using Lemma 3 ,

$$\mathcal{K}_p\left(\frac{1}{2}, \frac{1}{2}; 1\right) = \mathcal{J}_p\left(\frac{1}{2}, \frac{1}{2}; 1\right) \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{16},$$

and

$$\mathcal{J}_p\left(\frac{1}{2}, \frac{1}{2}; 1\right) = 1 + \frac{1}{p} + \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{k=1}^{\infty} \frac{\tau_4^2(p^k)}{p^k},$$

from which (34) follows.

Hence, up to error term of $O(Q^2 (\log Q)^{15+\epsilon})$

$$\begin{aligned} & \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, \chi\right) \right|^8 dy = 2(\mathcal{D}(\Psi, Q) + \mathcal{S}(\Psi, Q) + \mathcal{G}(\Psi, Q)) \\ &= 2 \left(\frac{2^{16} - 53524}{16!} \right) Q^2 (\log Q)^{16} \frac{\tilde{\Psi}(s)}{2} A(1/2) \\ & \quad \times \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{B_p(1/2)} \left(\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right)\right) \int_{-\infty}^{\infty} G(1/2, t) dt \\ &= \frac{24024}{16!} a_4 \sum_q \Psi\left(\frac{q}{Q}\right) \phi^b(q) (\log q)^{16} \prod_{p|q} B_p(1/2) \int_{-\infty}^{\infty} G(1/2, t) dt. \end{aligned}$$

This concludes the proof of Theorem 1.

11. PROOF OF PROPOSITION 3 – SHIFTED MOMENTS

In this section, we prove Proposition 3. As mentioned in the introduction, the proof of the Proposition is similar to the proof of Theorem 1.1 in [1], which gives upper bounds for shifted moments of the Riemann zeta function. We start by introducing some notation.

Given a real number $x \geq 0$ and a complex number z , we define

$$W = 2k^2 \log \log x + 2k^2 \min \left\{ \log \log x, \log \frac{1}{|z_1 + \bar{z}_2|} \right\}.$$

Note that W depends on z_1, z_2 , and x . Proposition 3 follows from the following proposition, which establishes an upper bound for the frequency of large values of

$$\log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k.$$

Proposition 7. *With assumptions as in Proposition 3, let $M(V; z_i, q)$ denote the number of even primitive characters $\chi \bmod q$ such that $\log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k \geq V$. For $10\sqrt{\log \log q} \leq V \leq W$ we have*

$$M(V; z_i, q) \ll \phi(q) \frac{V}{\sqrt{W}} \exp \left(-\frac{V^2}{W} \left(1 - \frac{4}{\log \log \log q} \right) \right).$$

When $W < V \leq \frac{1}{2}W \log \log \log q$, we have

$$M(V; z_i, q) \ll \phi(q) \frac{V}{\sqrt{W}} \exp \left(-\frac{V^2}{W} \left(1 - \frac{7V}{4W \log \log \log q} \right)^2 \right).$$

Finally, when $\frac{1}{2}W \log \log \log q < V$, we have

$$M(V; z_i, q) \ll \phi(q) \exp \left(-\frac{1}{129} V \log V \right).$$

Proof of Proposition 3. Summation by parts gives

$$\begin{aligned} \sum_{\chi \pmod{q}}^b \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^{2k} &= - \int_{-\infty}^{\infty} \exp(2V) dM(V; z_i, q) \\ &= 2 \int_{-\infty}^{\infty} \exp(2V) M(V; z_i, q) dV. \end{aligned}$$

By Proposition 7, $M(V; z_i, q) \ll \phi(q)(\log q)^\epsilon \exp \left(-\frac{V^2}{W} \right)$ for $3 \leq V \leq 4W$, and $M(V; z_i, q) \ll \phi(q)(\log q)^\epsilon \exp(-4V)$ for $V > 4W$. Inserting these bounds into the equation above, we obtain the Theorem 3. \square

We now focus on proving Proposition 7. We start with some preliminary results.

Lemma 10. *Let χ be a character mod q such that χ^2 is not the trivial character. Let z_i be defined as in Theorem 3. Then*

$$\begin{aligned} \log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k &\leq \operatorname{Re} \left(\sum_{p \leq x} \frac{\chi(p)}{p^{\left(\frac{1}{2} + \frac{1}{2 \log x} \right)}} \left(\sum_{i=1}^2 \frac{k}{p^{z_i}} \right) \frac{\log \frac{x}{p}}{\log x} \right) \\ &\quad + 3k \frac{\log q}{\log x} + O(\log_3 q). \end{aligned}$$

Proof. By modifying the proof of the Proposition in [16], we obtain that

$$\begin{aligned} \log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k &\leq \operatorname{Re} \left(\sum_{\substack{p^l \leq x \\ l \geq 1}} \frac{\chi(p^l)}{l p^{l(\frac{1}{2} + \frac{1}{2 \log x})}} \left(\sum_{i=1}^2 \frac{k}{p^{z_i}} \right) \frac{\log \frac{x}{p^l}}{\log x} \right) \\ &\quad + 3k \frac{\log q}{\log x} + O \left(\frac{1}{\log x} \right). \end{aligned}$$

When $l \geq 3$, the sum over p^l is absolutely convergent. Since χ^2 is not the trivial character, we have by GRH that

$$\sum_{p \leq y} \left(\sum_{i=1}^2 \frac{1}{p^{2z_i}} \right) \chi(p^2) \log p \ll \sqrt{y} (\log qy)^2.$$

The sum above is also trivially bounded $\ll y$. From these bounds and partial summation, we obtain

$$\operatorname{Re} \sum_{p \leq \sqrt{x}} \frac{\chi(p^2)}{2p^{1 + \frac{1}{\log x}}} \left(\sum_{i=1}^2 \frac{1}{p^{2z_i}} \right) \frac{\log \frac{x}{p^2}}{\log x} = O(\log_3 q).$$

□

Lemma 11. *Let q and y be real numbers and k a natural number with $y^k \leq \frac{\phi(q)}{\log q}$. For any complex numbers $a(p)$, we have*

$$\sum_{\chi \bmod q} \left| \sum_{p \leq y} \frac{a(p) \chi(p)}{p^{1/2}} \right|^{2k} \ll \phi(q) k! \left(\sum_{p \leq y} \frac{|a(p)|^2}{p} \right)^k,$$

where the implied constant is absolute.

Proof. The proof is similar to the proof of Lemma 5 in the introduction, but we use only the orthogonality of Dirichlet characters. □

Now we are ready to prove the Proposition.

Proof of Proposition 7. Set

$$A = \begin{cases} 2k \log \log \log q & V \leq W \\ \frac{2k}{V} W \log \log \log q & W < V \leq \frac{1}{2} W \log \log \log q \\ 4k & V > \frac{1}{2} W \log \log \log q. \end{cases}$$

Define $x = q^{A/V}$ and $z = x^{1/\log \log q}$. If we take $x = \frac{(\log q)^2}{\log \log q}$ in Lemma 10 and bound the sum over p trivially, we obtain that

$$\log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k \leq \frac{3k}{2} \frac{\log q}{\log \log q}$$

when q is large enough. Therefore we may assume that $10\sqrt{\log \log q} \leq V \leq \frac{3k \log q}{2 \log \log q}$. Then by Lemma 10 we have

$$\log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k \leq S_1 + S_2 + \frac{3kV}{A},$$

where S_1 is the sum truncated to $p \leq z$, and S_2 is the sum with $z < p \leq x$. If χ is a character such that

$$\log \left| L \left(\frac{1}{2} + z_1, \chi \right) L \left(\frac{1}{2} + z_2, \chi \right) \right|^k \geq V,$$

then either

$$S_2 \geq \frac{kV}{2A} \quad \text{or} \quad S_1 \geq V \left(1 - \frac{7k}{2A} \right) := V_1.$$

For any $\ell \leq \frac{V}{A} - 1$, by Lemma 11, we have

$$\sum_{\chi \bmod q}^b |S_2|^{2\ell} \ll \sum_{\chi \bmod q} |S_2|^{2\ell} \ll \phi(q) \ell! \left(\sum_{z < p \leq x} \frac{4k^2}{p} \right)^\ell \ll \phi(q) (4k^2 \ell \log_3 q)^\ell.$$

Moreover, for $\ell = \lfloor \frac{V}{A} - 1 \rfloor$, the number of even primitive characters $\chi \bmod q$ with $S_2 \geq kV/2A$ is bounded by

$$(35) \quad \phi(q) \left(\frac{2A}{kV} \right)^{2\ell} (4k^2 \ell \log_3 q)^\ell \ll \phi(q) \exp \left(-\frac{V}{2A} \log V \right).$$

Next, we consider the number of even primitive characters $\chi \bmod q$ such that $S_1 \geq V_1$. By Lemma 11, we find that for any $\ell \leq \frac{\log(\frac{\phi(q)}{\log q})}{\log z}$,

$$\sum_{\chi \bmod q}^b |S_1|^{2\ell} \leq \sum_{\chi \bmod q} |S_1|^{2\ell} \ll \phi(q) \ell! \left(\sum_{p \leq z} \frac{|a(p)|^2}{p} \right)^\ell,$$

where

$$a(p) = \frac{\log(x/p)}{p^{\frac{1}{2\log x}} \log x} (p^{-z_1} + p^{-z_2}).$$

Assuming GRH, we have

$$\sum_{p \leq y} \left(\sum_{i=1}^2 p^{-2z_i} \right) \log p = \sum_{i=1}^2 \frac{y^{1-2z_i}}{1-2z_i} + O(\sqrt{y}(\log qy)^2).$$

Also the sum is trivially bounded by y . By partial summation and the bounds above, we obtain

$$\sum_{p \leq z} \frac{|a(p)|^2}{p} \leq \sum_{p \leq q} \frac{1}{p} |p^{-z_1} + p^{-z_2}|^2 = W + O(\log_3 q).$$

Hence the number of even primitive characters $\chi \bmod q$ such that $S_1 \geq V_1$ is

$$(36) \quad \ll \phi(q) V_1^{-2\ell} \ell! (W + O(\log_3 q)) \ll \phi(q) \sqrt{\ell} \left(\frac{\ell(W + O(\log_3 q))}{eV_1^2} \right)^\ell.$$

When $V \leq (\log \log q)^2$, we take $\ell = \lfloor \frac{V_1^2}{W} \rfloor$, and for $V > (\log \log q)^2$, we take $\ell = \lfloor 10V \rfloor$. Therefore the number of even primitive characters $\chi \bmod q$ such that $S_1 \geq V_1$ is

$$\ll \phi(q) \frac{V}{\sqrt{W}} \exp \left(-\frac{V_1^2}{W} \right) + \phi(q) \exp(-4V \log V).$$

Combining the estimates (35) and (36) gives the proposition. □

12. THE FOURTH MOMENT OF DIRICHLET TWISTS OF A $GL(2)$ AUTOMORPHIC L -FUNCTIONS

In this section, we will discuss how to modify the proof of Theorem 1 to prove Theorem 2. The proof may be carried in the same manner, and indeed is slightly easier as the main term contribution comes solely from the diagonal term.

Define

$$G_f(s, t) := \Gamma^2\left(\frac{k-1}{2} + s + it\right) \Gamma^2\left(\frac{k-1}{2} + s - it\right); \quad \sigma_f(n) = \sum_{n=n_1 n_2} a(n_1) a(n_2);$$

$$W_f(x, t) := \frac{1}{2\pi i} \int_{(1)} G_f(1/2 + s, t) x^{-s} \frac{ds}{s};$$

$$V_f(\xi, \eta; \mu) = \int_{-\infty}^{\infty} \left(\frac{\eta}{\xi}\right)^{it} W_f\left(\frac{\xi \eta (2\pi)^4}{\mu^4}, t\right) dt;$$

and

$$\Lambda_1(f \times \chi) = \sum_{m, n=1}^{\infty} \frac{\sigma_f(m) \sigma_f(n)}{\sqrt{mn}} \chi(m) \overline{\chi}(n) V_f(m, n; q).$$

By using the functional equation in (3) and following the same arguments as Lemma 1, we can write

$$\mathcal{M}_f = \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^* \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, f \times \chi\right) \right|^4 dy = 2\Delta_f(\Psi, Q),$$

where

$$\Delta_f(\Psi, Q) = \sum_q \sum_{\chi \pmod{q}}^* \Psi\left(\frac{q}{Q}\right) \Lambda_1(f \times \chi).$$

By orthogonality relation of \sum^* in Lemma 5, we obtain that

$$\Delta_f(\Psi, Q) = \sum_{m, n=1}^{\infty} \frac{\sigma_f(m) \sigma_f(n)}{\sqrt{mn}} \sum_{\substack{d, r \\ (dr, mn)=1 \\ r|m-n}} \phi(r) \mu(d) \Psi\left(\frac{dr}{Q}\right) V_f(m, n, dr).$$

By the same arguments as Section 3, we can truncate the sum with acceptable error term by invoking Proposition 4. Therefore we consider

$$\tilde{\Delta}_f(\Psi, Q) = \sum_{m, n=1}^{\infty} \frac{\sigma_f(m) \sigma_f(n)}{\sqrt{mn}} \sum_{\substack{d, r \\ (dr, mn)=1 \\ r|m-n}} \phi(r) \mu(d) \Psi\left(\frac{dr}{Q}\right) V_f\left(m, n, \frac{dr}{(\log Q)^\alpha}\right),$$

We can write

$$\tilde{\Delta}_f(\Psi, Q) = \mathcal{D}_f(\Psi, Q) + \mathcal{S}_f(\Psi, Q) + \mathcal{G}_f(\Psi, Q)$$

where the diagonal term $\mathcal{D}_f(\Psi, Q)$ consists of the terms with $m = n$, the term $\mathcal{S}_f(\Psi, Q)$ consists of the remaining terms with $d > D$ and $\mathcal{G}_f(\Psi, Q)$ consists of the rest of the terms with $d < D$.

Since $L(s, f \times \chi_0)$ is an entire function, we can treat $L(s, f \times \chi_0)$ in the same way as treating non-principle character twists. Therefore, we bound $\mathcal{S}_f(\Psi, Q)$ and $\mathcal{G}_f(\Psi, Q)$ in the same way as bounding non-principal character contributions of $\mathcal{S}(\Psi, Q)$ (Section 5) and $\mathcal{G}(\Psi, Q)$ (Section 7 and using Proposition 2). Moreover, there is no main term contribution from off-diagonal terms.

For the diagonal term $\mathcal{D}_f(\Psi, Q)$, the argument is the same as in Section 4.1, but we use Lemma 4 instead. This completes the proof of Theorem 2.

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